A SPECTRAL NAKAYAMA'S LEMMA

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Definition 0.1. Let $f : \widetilde{R} \to R$ be a square-zero extension of $\mathbb{E}_\infty$-rings, and let N be an R-module. A *deformation* of N along f is an $\widetilde{R}$-module $\widetilde{N}$ such that $R \otimes_{\widetilde{R}} \widetilde{N} \simeq N$.

Theorem 0.2 (Nakayama's Lemma, square-zero version)**.** *Let $\widetilde{R} \to R$ be a square-zero extension of $\mathbb{E}_\infty$-rings, and let N be an $\widetilde{R}$-module such that $R \otimes_{\widetilde{R}} N = 0$. Then $N = 0$.*

In other words, the zero module deforms uniquely along square-zero extensions.

Proof. We have a fiber sequence of $\widetilde{R}$-modules

$$M \to \widetilde{R} \to R$$

for some $M \in \mathrm{Mod}_{\widetilde{R}}$. Tensoring with N, we get

$$M \otimes_{\widetilde{R}} N \to N \to 0,$$

so $N \simeq M \otimes_{\widetilde{R}} N$.

On the other hand, we can instead tensor our fiber sequence with M to get

$$M \otimes_{\widetilde{R}} M \to M \to M \otimes_{\widetilde{R}} R.$$

By Proposition 7.4.1.14 of [1], the first map in this sequence is 0. Consequently, we obtain a splitting

$$M \otimes_{\widetilde{R}} R \simeq M \oplus \Sigma(M \otimes_{\widetilde{R}} M).$$

Tensoring this equivalence with N gives

$$M \otimes_{\widetilde{R}} R \otimes_{\widetilde{R}} N \simeq (M \otimes_{\widetilde{R}} N) \oplus \Sigma(M \otimes_{\widetilde{R}} M \otimes_{\widetilde{R}} N);$$

but the left side of this equivalence is 0 since $R \otimes_{\widetilde{R}} N = 0$, and the first summand on the right side is N. It follows that $N = 0$. □

Definition 0.3. We say a morphism $\widetilde{R} \to R$ of $\mathbb{E}_\infty$-rings is a *nilpotent extension* if it can be written as a finite composite of square-zero extensions. We say it is an *infinitesimal* extension if it can be written as a possibly infinite composite of square-zero extensions; that is to say, it is the limit of a possibly infinite sequence of square-zero extensions.

Corollary 0.4. *The zero module deforms uniquely along nilpotent extensions.*

Proof. Apply Theorem 0.2 and induct. □

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References

[1] Jacob Lurie. *Higher Algebra*. Sept. 18, 2017. URL: https://www.math.ias.edu/~lurie/papers/HA.pdf.